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The polarization tensor for a magnetized vacuum

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Abstract. The exact vacuum polarization 4-tensor in the presence of a static magnetic field is calculated by a method due to Svetozarova and Tsytovich. The Hermitian part of the tensor is explicitly renormalized by Shabad's diagonalization with respect to tensor indices and the anti-Hermitian part of the tensor is found to be finite and gauge invariant. An alternative method for calculating the vacuum polarization tensor is developed using the relation between the anti-Hermitian part of the tensor and the probability of emission and absorption of a photon by an electron in a static magnetic field.

1. Introduction

In an earlier paper (Melrose and Stoneham 1976, referred to as I) we presented an exact calculation of the polarization tensor for a magnetized vacuum and we used this to calculate exact refractive indices. The important ingredients in the calculations in I were the use of the exact electron propagator in the form due to Géhéniau (1950) and Géhéniau and Demeur (1951), a simple method, based on an idea due to Shabad (1975), for renormalizing (regularizing) the tensor, and the use of the methods of plasma physics to derive the wave properties. The resulting expressions are exact in the sense that unlike other calculations, e.g. Toll (1952), Adler (1971), Constantinescu (1972a, b), Tsai and Erber (1974) and Shabad (1975), neither the dipole approximation nor the weak-field approximation is made. Furthermore, the refractive indices are not assumed to be approximately equal to unity but are found exactly.

In this paper we present an alternative form of the polarization tensor for a magnetized vacuum. The basic calculation is a minor generalization of one presented by Svetozarova and Tsytovich (1962), who calculated the unrenormalized 3-tensor but did not renormalize it. We calculate the unrenormalized 4-tensor by the same method and renormalize it by the method of I. The result appears qualitatively different from the result found in I due to the different forms of the propagator used in the two calculations. (Only in special cases have we been able to show explicitly that the two forms of the tensor are equivalent, cf appendix 2.) The alternative form derived here is convenient for separating into Hermitian and anti-Hermitian parts, and, more generally, is in a form familar in plasma physics. In fact Svetozarova and Tsytovich (1962) and Melrose (1974a) have used the present approach to derive a relativistic quantum expression for the dielectric tensor of an electron gas.

From another viewpoint, we indicate how the anti-Hermitian part of the tensor may be related to known expressions for the probability of emission and absorption of a photon by an electron in a magnetostatic field, and how the Kramers–Kronig relations may be used to derive the Hermitian part of the tensor. The implied method for calculating the vacuum polarization tensor is a generalization of Toll's (1952) method for calculating the refractive indices of a magnetized vacuum, and it is also a generalization of a well known method for an unmagnetized vacuum (e.g. Euwema and Wheeler 1956, Lifshitz and Pitaevskii 1974, § 110). This alternative method reproduces the result we find by direct calculation using the approach outlined above.

In § 2 we write down the Hermitian part of the unrenormalized polarization 4-tensor for a magnetized vacuum; the 3-tensor part is that calculated by Svetozarova and Tsytovich (1962). We simplify the tensor by performing some of the indicated summations explicitly. The tensor is renormalized in § 3 using the method presented in I. In § 4 we calculate the exact anti-Hermitian part of the vacuum polarization 4-tensor which may be used to discuss gyromagnetic absorption and absorption due to pair creation. The threshold photon energies for the anti-Hermitian part of the tensor to be non-zero are shown to include those corresponding to the limiting energies for pair production as given by Toll (1952). Section 5 contains the alternative derivation of the vacuum polarization tensor based on Toll's method and on the well known method for the zero-field vacuum polarization tensor. In the strong-field limit and in the dipole approximation our results reproduce those of Tsai and Erber (1974) and Heisenberg and Euler (1936), respectively. For the anti-Hermitian part of the tensor we correct the result of Tsytovich (1961) for the zero-field tensor.

It is worth emphasizing that we describe the response of the medium (a magnetized vacuum) as is done conventionally in plasma physics where one incorporates both the electric and magnetic responses of the medium in an equivalent dielectric tensor. This tensor is related to the vacuum polarization tensor by

$$\varepsilon_{ij}(\boldsymbol{k},\omega)=\delta_{ij}+\frac{4\pi}{\omega^2}\alpha_{ij}(\boldsymbol{k},\omega), \tag{1}$$

with $\alpha_{ij}(\boldsymbol{k},\omega)$ equal to minus the $\mu=i$, $\nu=j$ components of the renormalized vacuum polarization tensor, reg $\alpha^{\mu\nu}(\boldsymbol{k},\omega)$. The separation into electric and magnetic responses may be obtained by writing down the induced current and expanding it in multipole moments, but this procedure is useful only when moments other than the electric and magnetic dipoles and electric quadrupole may be neglected.

Our notation is that of Berestetskii *et al* (1971) (but with $-e$ for the electronic charge) and unrationalized Gaussian units with $\hbar=c=1$ are used. The symbols := and =: define the quantities on the left and right respectively and $A^{\mu}=(A^0,\boldsymbol{A})$ relates a 4-vector to its time and space components and $\boldsymbol{A}=(A_1,A_2,A_3)$ relates the 3-vector to its Cartesian components.

2. The Hermitian part of the unrenormalized vacuum polarization tensor

The Hermitian part of the unrenormalized vacuum polarization 3-tensor in the presence of a static magnetic field was calculated by Svetozarova and Tsytovich (1962) and the details of the calculation were given by Melrose (1974a, appendix A). Here we extend this calculation to find the corresponding 4-tensor $\alpha^{\mu\nu}(\boldsymbol{k},\omega)$. The Hermitian part of the tensor is given by

$$\alpha^{\mu\nu(\mathrm{H})} = -\frac{e^3 B}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}p_z}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}p'_z}{2\pi}2\pi\delta(p'_z - p_z + k_\|)$$
$$\times\sum_{\substack{n,n'\\ s,s'}}\sum_{\epsilon,\epsilon'}\frac{(\epsilon'-\epsilon)(\Gamma^{\epsilon\epsilon'}_{qq'})^\mu[(\Gamma^{\epsilon\epsilon'}_{qq'})^\nu]^*}{2(\omega+\epsilon'\varepsilon_{q'}-\epsilon\varepsilon_q)}, \tag{2}$$

where * denotes complex conjugation and where the sums are over $n = 0, 1, 2, \ldots$; $n' = 0, 1, 2, \ldots$; $s = \pm 1$, $s' = \pm 1$, $\epsilon = \pm 1$, $\epsilon' = \pm 1$, with

$$\begin{aligned} \varepsilon_q &:= [m^2 + p_z^2 + (2n+1+s)eB]^{1/2},\\ \varepsilon_{q'} &:= [m^2 + (p'_z)^2 + (2n'+1+s')eB]^{1/2}. \end{aligned} \tag{3}$$

With the coordinate axes defined by

$$\begin{aligned} \boldsymbol{k} &= (k_x, 0, k_\|) =: ((\operatorname{sgn} k_x)k_\perp, 0, k_\|),\\ \boldsymbol{B} &= (0, 0, B), \end{aligned} \tag{4}$$

one has

$$(\Gamma^{\epsilon\epsilon'}_{qq'})^\mu = \frac{1}{\sqrt{(4\epsilon\epsilon'\varepsilon_q\varepsilon_{q'})}}\left\{\frac{(1+ss')}{2}\left[\operatorname{sgn} k_x\left(tJ^n_\nu + \frac{1}{t}(p_z p'_z J^n_\nu + p_\perp p'_\perp J^{n+s}_\nu)\right),\right.\right.$$
$$-\left(rp'_\perp J^n_{\nu+s} + \frac{1}{r}p_\perp J^{n+s}_{\nu+s}\right), -\mathrm{i}s\left(rp'_\perp J^n_{\nu+s} - \frac{1}{r}p_\perp J^{n+s}_{\nu-s}\right), \left.\operatorname{sgn} k_x\left(rp'_z + \frac{1}{r}p_z\right)J^n_\nu\right]$$
$$+\frac{(1-ss')}{2}\left[\frac{s}{t}\operatorname{sgn} k_x(p'_z p_\perp J^{n+s}_{\nu-s} - p_z p'_\perp J^n_{\nu-s}), -s\left(rp'_z - \frac{1}{r}p_z\right)J^n_\nu,\right.$$
$$\left.\left.-\mathrm{i}\left(rp'_z - \frac{1}{r}p_z\right)J^n_\nu, -s\operatorname{sgn} k_x\left(rp'_\perp J^n_{\nu-s} - \frac{1}{r}p_\perp J^{n+s}_{\nu-s}\right)\right]\right\}, \tag{5}$$

with

$$\begin{aligned} \nu &:= n' - n,\\ r &:= \left(\frac{\epsilon\varepsilon_q + m}{\epsilon'\varepsilon_{q'} + m}\right)^{1/2},\\ t &:= [(\epsilon\varepsilon_q + m)(\epsilon'\varepsilon_{q'} + m)]^{1/2},\\ p_\perp &:= [(2n+1+s)eB]^{1/2},\\ p'_\perp &:= [(2n'+1+s')eB]^{1/2}, \end{aligned} \tag{6}$$

$$J^n_\nu(x) := \left(\frac{n!}{(n+\nu)!}\right)^{1/2}\exp(-\tfrac{1}{2}x)x^{\nu/2}L^\nu_n(x) = (-1)^\nu J^{n+\nu}_{-\nu}(x), \tag{7}$$

where L^ν_n is a generalized Laguerre polynomial in the notation of Gradsteyn and Ryzhik (1965). The argument of the functions J^n_ν, etc, in (5) is $k_\perp^2/2eB$.

The foregoing expression for $\alpha^{\mu\nu(\mathrm{H})}(\boldsymbol{k}, \omega)$ corrects that given by Melrose (1974a) which had the incorrect sign attached to $(\Gamma^{\epsilon\epsilon'}_{qq'})^3$. It includes the sign of k_x explicitly to facilitate discussion of the symmetry properties of the tensor.

We now proceed to simplify the expression (2) for the unrenormalized vacuum polarization tensor by performing the sums over ϵ, ϵ', s and s' explicitly. Firstly, note that the factor $(\epsilon - \epsilon')$ in (2) implies that the only terms which contribute have $\epsilon' = -\epsilon$. Consequently (2) may be rewritten in the form

$$\alpha^{\mu\nu(\mathrm{H})}(\boldsymbol{k}, \omega) = \frac{e^3 B}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}p_z}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}p'_z}{2\pi}\, 2\pi\delta(p'_z - p_z + k_\|)\sum_{\substack{n,n'\\ s,s'}}\sum_{\epsilon}\frac{\epsilon(\Gamma^{\epsilon,-\epsilon}_{qq'})^{\mu}[(\Gamma^{\epsilon,-\epsilon}_{qq'})^{\nu}]^*}{\omega - \epsilon(\varepsilon_q + \varepsilon_{q'})}. \tag{8}$$

Further simplification is achieved by re-labelling the dummy summation and integration variables in (8) (with (5)), specifically $p_z \leftrightarrow p'_z$, $s \leftrightarrow s'$, $n \leftrightarrow n'$ (implying $\nu \leftrightarrow -\nu$), and then interchanging the orders of integration. Making the replacements $p_z \to -p_z$, $p'_z \to -p'_z$ and $\epsilon \to -\epsilon$ then gives, as an alternative to (8),

$$\alpha^{\mu\nu(\mathrm{H})}(\boldsymbol{k}, \omega) = -\frac{e^3 B}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}p_z}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}p'_z}{2\pi}\, 2\pi\delta(p'_z - p_z + k_\|)\sum_{\substack{n,n'\\ s,s'}}\sum_{\epsilon}\frac{\epsilon f(\mu,\nu)(\Gamma^{\epsilon,-\epsilon}_{qq'})^{\mu}[(\Gamma^{\epsilon,-\epsilon}_{qq'})^{\nu}]^*}{\omega + \epsilon(\varepsilon_q + \varepsilon_{q'})}, \tag{9}$$

with

$$f(\mu,\nu) := \begin{cases} 1, & \text{for } \mu\nu = 00, 11, 22, 33, 02, 13; \\ -1, & \text{for } \mu\nu = 01, 03, 12, 23, \end{cases}$$

and where we have used the relation (7). Half the sum of the two forms (8) and (9) gives

$$\alpha^{\mu\nu(\mathrm{H})}(\boldsymbol{k}, \omega) = \frac{e^3 B}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}p_z}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}p'_z}{2\pi}\, 2\pi\delta(p'_z - p_z + k_\|)\sum_{\substack{n,n',\\ s,s'}}\sum_{\epsilon}\frac{g(\mu,\nu)(\Gamma^{\epsilon,-\epsilon}_{qq'})^{\mu}[(\Gamma^{\epsilon,-\epsilon}_{qq'})^{\nu}]^*}{\omega^2 - (\varepsilon_q + \varepsilon_{q'})^2}, \tag{10}$$

with

$$g(\mu,\nu) = \begin{cases} \varepsilon_q + \varepsilon_{q'}, & \text{for } \mu\nu = 00, 11, 22, 33, 02, 13; \\ \epsilon\omega, & \text{for } \mu\nu = 01, 03, 12, 23. \end{cases} \tag{11}$$

The sums over ϵ and over s and s' in (10) may now be performed. Introducing the shorthand notations

$$\begin{aligned} \varepsilon_n &:= (m^2 + p_z^2 + 2neB)^{1/2}, & p_n &:= (2neB)^{1/2}, \\ \varepsilon_{n'} &:= (m^2 + (p'_z)^2 + 2n'eB)^{1/2}, & p_{n'} &:= (2n'eB)^{1/2}, \end{aligned} \tag{12}$$

$$a_n = \begin{cases} 2 & \text{for } n \geq 1; \\ 1 & \text{for } n = 0, \end{cases} \tag{13}$$

and using the relations (A.1) and the identities

$$(J^{n+1}_{-n-1})^2 = (J^0_{n+1})^2, \qquad (J^n_{-n})^2 = (J^0_n)^2, \qquad J^0_n J^0_{n+1} = -J^n_{-n} J^{n+1}_{-n-1}, \tag{14}$$

gives

$$\alpha^{\mu\nu(\mathrm{H})}(\boldsymbol{k},\omega)=-\frac{e^3B}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}p_z}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}p'_z}{2\pi}2\pi\delta(p'_z-p_z+k_\|)$$
$$\times\left(\sum_{n,n'}\frac{d_1^{\mu\nu}}{[\omega^2-(\varepsilon_{n+1}+\varepsilon_{n'+1})^2]\varepsilon_{n+1}\varepsilon_{n'+1}}+\sum_{n=0}^{\infty}d_2^{\mu\nu}\right),$$

with

$$d_1^{00}=(\varepsilon_{n+1}+\varepsilon_{n'+1})\{(m^2+p_zp'_z-\varepsilon_{n+1}\varepsilon_{n'+1})[(J_\nu^n)+(J_\nu^{n+1})^2]+2p_{n+1}p_{n'+1}J_\nu^nJ_\nu^{n+1}\},$$
$$d_1^{11}=(\varepsilon_{n+1}+\varepsilon_{n'+1})\{(-m^2-p_zp'_z-\varepsilon_{n+1}\varepsilon_{n'+1})[(J_{\nu+1}^n)^2+(J_{\nu-1}^{n+1})^2]+2p_{n+1}p_{n'+1}J_{\nu+1}^nJ_{\nu-1}^{n+1}\},$$
$$d_1^{22}=(\varepsilon_{n+1}+\varepsilon_{n'+1})\{(-m^2-p_zp'_z-\varepsilon_{n+1}\varepsilon_{n'+1})[(J_{\nu+1}^n)^2+(J_{\nu-1}^{n+1})^2]-2p_{n+1}p_{n'+1}J_{\nu+1}^nJ_{\nu-1}^{n+1}\},$$
$$d_1^{33}=(\varepsilon_{n+1}+\varepsilon_{n'+1})\{(-m^2+p_zp'_z-\varepsilon_{n+1}\varepsilon_{n'+1})[(J_\nu^n)^2+(J_\nu^{n+1})^2]-2p_{n+1}p_{n'+1}J_\nu^nJ_\nu^{n+1}\},$$
$$d_1^{01}=(\operatorname{sgn}k_x)\omega[\varepsilon_{n'+1}p_{n+1}(J_{\nu+1}^nJ_\nu^{n+1}+J_{\nu-1}^{n+1}J_\nu^n)-\varepsilon_{n+1}p_{n'+1}(J_\nu^nJ_{\nu+1}^n+J_\nu^{n+1}J_{\nu-1}^{n+1})],$$
$$d_1^{03}=\omega(\varepsilon_{n+1}p'_z-\varepsilon_{n'+1}p_z)[(J_\nu^n)^2+(J_\nu^{n+1})^2]$$
$$d_1^{13}=-\operatorname{sgn}k_x(\varepsilon_{n+1}+\varepsilon_{n'+1})[p_{n'+1}p_z(J_{\nu+1}^nJ_\nu^n+J_{\nu-1}^{n+1}J_\nu^{n+1})+p_{n+1}p'_z(J_{\nu+1}^nJ_\nu^{n+1}+J_{\nu-1}^{n+1}J_\nu^n)],$$

$$\begin{aligned}
&d_1^{02}=d_1^{12}=d_1^{23}=0,\\
&d_2^{00}=a_n\frac{(\varepsilon_n+\varepsilon_{0'})(J_n^0)^2(m^2+p_zp'_z-\varepsilon_n\varepsilon_{0'})}{[\omega^2-(\varepsilon_n+\varepsilon_{0'})^2]\varepsilon_n\varepsilon_{0'}},\\
&d_2^{11}=d_2^{22}=\frac{2(\varepsilon_{n+1}+\varepsilon_{0'})(J_n^0)^2(-m^2-p_zp'_z-\varepsilon_{n+1}\varepsilon_{0'})}{[\omega^2-(\varepsilon_{n+1}+\varepsilon_{0'})^2]\varepsilon_{n+1}\varepsilon_{0'}},\\
&d_2^{33}=a_n\frac{(\varepsilon_n+\varepsilon_{0'})(J_n^0)^2(-m^2+p_zp'_z-\varepsilon_n\varepsilon_{0'})}{[\omega^2-(\varepsilon_n+\varepsilon_{0'})^2]\varepsilon_n\varepsilon_{0'}}\\
&d_2^{01}=\frac{-2\omega(\operatorname{sgn}k_x)p_nJ_{n-1}^0J_n^0}{[\omega^2-(\varepsilon_n+\varepsilon_{0'})^2]\varepsilon_n}\\
&d_2^{03}=a_n\frac{\omega(\varepsilon_np'_z-\varepsilon_{0'}p_z)(J_n^0)^2}{[\omega^2-(\varepsilon_n+\varepsilon_{0'})^2]\varepsilon_n\varepsilon_{0'}}\\
&d_2^{13}=\frac{2p_np'_z\operatorname{sgn}k_x(\varepsilon_n+\varepsilon_{0'})J_{n-1}^0J_n^0}{[\omega^2-(\varepsilon_n+\varepsilon_{0'})^2]\varepsilon_n\varepsilon_{0'}}\\
&d_2^{02}=d_2^{12}=d_2^{23}=0.
\end{aligned}\tag{15}$$

This completes our simplification of the Hermitian part of the unrenormalized vacuum polarization tensor.

3. Renormalization

The vacuum polarization tensor may be renormalized by the method developed in I. The essence of the method is that the renormalized tensor must be of the form

$$\operatorname{reg}\alpha^{\mu\nu}(\boldsymbol{k},\omega)=\sum_{i=0}^{2}G_if_i^{\mu\nu},\tag{16}$$

where the G_i are functions only of the invariants B^2, $\omega^2-k_\parallel^2$ and $k_\perp^2$, with

$$f_0^{\mu\nu} := g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}$$
$$f_i^{\mu\nu} := \frac{b_i^\mu b_i^\nu}{(b_i)^2} \qquad (i=1,2), \tag{17}$$

and with

$$b_1^\mu := F^{\mu\rho}k_\rho,$$
$$b_2^\mu := -\tfrac{1}{2}\varepsilon^{\mu\rho\sigma\tau}F_{\sigma\tau}k_\rho =: F^{*\mu\rho}k_\rho. \tag{18}$$

(The Hermitian part of the tensor is given by (16) with only the real parts of the G_i retained.) The G_i may be found by equating them to the components of the unrenormalized tensor along the $f_i^{\mu\nu}$. These components are necessarily finite. The remaining parts of the unrenormalized tensor are the gauge-dependent and divergent parts; that is, the remaining parts are just those discarded in any renormalization procedure.

Explicit calculation gives

$$\operatorname{Re} G_i = -\frac{e^3B}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}p_z}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}p_z'}{2\pi}2\pi\delta(p_z'-p_z+k_\parallel)$$
$$\times\left(\sum_{n,n'}\frac{g_i}{[\omega^2-(\varepsilon_{n+1}+\varepsilon_{n'+1})^2]\varepsilon_{n+1}\varepsilon_{n'+1}}+\sum_{n=0}^{\infty}\frac{a_nh_i}{[\omega^2-(\varepsilon_n+\varepsilon_{0'})^2]\varepsilon_n\varepsilon_{0'}}\right),$$

where Re denotes the real part, with

$$g_0 = -\frac{2k^2}{k_x}\varepsilon_{n'+1}p_{n+1}(J_{\nu+1}^nJ_\nu^{n+1}+J_\nu^nJ_{\nu-1}^{n+1}),$$
$$g_1 = 4(\varepsilon_{n+1}+\varepsilon_{n'+1})p_{n+1}p_{n'+1}J_{\nu+1}^nJ_{\nu-1}^{n+1}-2k_x\varepsilon_{n'+1}p_{n+1}(J_{\nu+1}^nJ_\nu^{n+1}+J_\nu^nJ_{\nu-1}^{n+1}),$$
$$g_2 = \frac{(\omega^2-k_\parallel^2)}{k_xk_\parallel}\{2k_\parallel\varepsilon_{n'+1}p_{n+1}(J_{\nu+1}^nJ_\nu^{n+1}+J_\nu^nJ_{\nu-1}^{n+1})$$
$$-k_x(\varepsilon_{n+1}p_z'-\varepsilon_{n'+1}p_z)[(J_\nu^n)^2+(J_\nu^{n+1})^2]\}; \tag{19}$$
$$h_0 = k^2\operatorname{sgn}k_x(J_{n-1}^0)^2\varepsilon_{0'},$$
$$h_1 = k_\perp^2\operatorname{sgn}k_x(J_{n-1}^0)^2\varepsilon_{0'},$$
$$h_2 = -(\omega^2-k_\parallel^2)\left(\operatorname{sgn}k_x(J_{n-1}^0)^2\varepsilon_{0'}+\frac{(\varepsilon_np_z'-\varepsilon_{0'}p_z)}{k_\parallel}(J_n^0)^2\right).$$

The renormalized vacuum polarization tensor given by (16) (with (17), (18) and (19)) must be equivalent to that obtained in I, but it is obviously in quite a different form.

4. The anti-Hermitian part of the vacuum polarization tensor

The anti-Hermitian part of the vacuum polarization tensor, $\alpha^{\mu\nu(\mathrm{a})}(\boldsymbol{k},\omega)$, may be obtained from the Hermitian part (2) (or (15)) by replacing ω by $\omega+\mathrm{i}0$ and retaining only the semi-residues resulting from the application of the Plemelj formula (Montgomery and Tidman 1964, § 5.3). The delta functions obtained by this procedure imply relations of the form

$$|\omega| = \varepsilon_{n+1} + \varepsilon_{n'+1}. \tag{20}$$

Equation (20) has solutions $p_z = p_z^+$ and $p_z = p_z^-$, with

$$p_z^\pm := \tfrac{1}{2}k_\parallel \beta_\nu^- \pm \tfrac{1}{2}|\omega|\kappa_{\nu,n}, \qquad \beta_\nu^\pm := 1 \pm \frac{2\nu eB}{(\omega^2 - k_\parallel^2)}, \tag{21}$$

and

$$\kappa_{\nu,n} := \left((\beta_\nu^-)^2 - \frac{4(m^2 + p_{n+1}^2)}{(\omega^2 - k_\parallel^2)}\right)^{1/2}. \tag{22}$$

The integral over p_z in the anti-Hermitian part derived using (15) is then trivial. Since the polarization tensor is symmetric, the anti-Hermitian part is i times the imaginary part. (It is convenient to introduce unit step functions $\theta(\kappa_{\nu,n}^2)$ when writing down the components of $\alpha^{\mu\nu(\mathrm{a})}(\boldsymbol{k}, \omega)$.)

The requirement that $\kappa_{\nu,n}$ be real implies a restriction on ω, specifically $\omega^2 < \omega_-^2$ or $\omega^2 > \omega_+^2$, with

$$\omega_\pm^2 := k_\parallel^2 + 2(m^2 + p_{n+1}^2 + \nu eB) \pm 2[(m^2 + p_{n+1}^2)(m^2 + p_{n'+1}^2)]^{1/2}. \tag{23}$$

Alternatively the requirement may be expressed as a limit on the values of n and n' which are to be included in the summations (cf appendix 3). The equation $|\omega| = \varepsilon_{n+1} - \varepsilon_{n'+1}$ also has solutions $p_z = p_z^+$ and $p_z = p_z^-$. This equation is the condition for gyromagnetic absorption by electrons or positrons to be possible. Gyromagnetic absorption does not contribute to $\alpha^{\mu\nu(\mathrm{a})}(\boldsymbol{k}, \omega)$ in a vacuum. Only the absorption due to pair creation is to be included for the vacuum. According to Svetozarova and Tsytovich (1962) only the solutions of (20) with $\omega^2 > \omega_+^2$ correspond to pair creation and only these are to be retained.

By applying the foregoing procedure to (15) one obtains

$$\alpha^{\mu\nu(\mathrm{a})}(\boldsymbol{k}, \omega) = \frac{\mathrm{i}e^3 B \operatorname{sgn}\omega}{4\pi|\omega^2 - k_\parallel^2|}\left(\sum_{n,n'} \frac{b_1^{\mu\nu}\theta(\kappa_{\nu,n}^2)}{\kappa_{\nu,n}} + \sum_{n=0}^{\infty} b_2^{\mu\nu}\right),$$

with

$$\begin{aligned}
b_1^{11} &= \{[p_{n+1}^2 + p_{n'+1}^2 - (\omega^2 - k_\parallel^2)][(J_{\nu+1}^n)^2 + (J_{\nu-1}^{n+1})^2] + 4p_{n+1}p_{n'+1}J_{\nu+1}^n J_{\nu-1}^{n+1}\},\\
b_1^{22} &= \{[p_{n+1}^2 + p_{n'+1}^2 - (\omega^2 - k_\parallel^2)][(J_{\nu+1}^n)^2 + (J_{\nu-1}^{n+1})^2] - 4p_{n+1}p_{n'+1}J_{\nu+1}^n J_{\nu-1}^{n+1}\},\\
b_1^{33} &= \{[-2m^2 + \tfrac{1}{2}(\omega^2 + k_\parallel^2)(\kappa_{\nu,n}^2 - \beta_\nu^+\beta_\nu^-)][(J_\nu^n)^2 + (J_\nu^{n+1})^2] - 4p_{n+1}p_{n'+1}J_\nu^n J_\nu^{n+1}\},\\
b_2^{11} &= b_2^{22} = -2(\omega^2 - k_\parallel^2)(J_n^0)^2\beta_{-n-1}^+ \frac{\theta(\kappa_{-n-1,n}^2)}{\kappa_{-n-1,n}},\\
b_2^{33} &= a_n(J_n^0)^2[-2m^2 + \tfrac{1}{2}(\omega^2 + k_\parallel^2)(\kappa_{-n,n-1}^2 - \beta_{-n}^+\beta_{-n}^-)]\frac{\theta(\kappa_{-n,n-1}^2)}{\kappa_{-n,n-1}},
\end{aligned} \tag{24}$$

and with the remaining components of $\alpha^{\mu\nu(\mathrm{a})}(\boldsymbol{k}, \omega)$ determined by the requirements of gauge invariance. Note that we do not impose gauge invariance, the result which emerges is gauge invariant. To establish the gauge invariance of $\alpha^{\mu\nu(\mathrm{a})}(\boldsymbol{k}, \omega)$ we needed the identities

$$\begin{aligned} k_\perp(J^{n+1}_{\nu-1}J^{n+1}_{\nu}+J^{n}_{\nu}J^{n}_{\nu+1})+2p_{n+1}J^{n}_{\nu}J^{n+1}_{\nu}&=p_{n'+1}[(J^{n}_{\nu})^2+(J^{n+1}_{\nu})^2], \\ (p^2_{n'+1}-p^2_{n+1})(J^{n}_{\nu+1}J^{n+1}_{\nu}+J^{n}_{\nu}J^{n+1}_{\nu-1})&=k_\perp p_{n+1}[(J^{n}_{\nu+1})^2+(J^{n+1}_{\nu-1})^2]+2k_\perp p_{n'+1}J^{n}_{\nu+1}J^{n+1}_{\nu-1}, \end{aligned} \tag{25}$$

as well as identities obtained from these by making the replacements $n \leftrightarrow n'$ and using (7). A separation of $\alpha^{\mu\nu(\mathrm{a})}(\boldsymbol{k}, \omega)$ into components along the $f_i^{\mu\nu}$ defined by (17) gives the imaginary parts of the G_i.

The presence of the functions $\theta(\kappa^2_{0,-1})$ and $\theta(\kappa^2_{1,-1})$ in (24) implies that absorption due to pair production is only possible for $\omega^2 > k_\parallel^2 + 4m^2$ and $\omega^2 > k_\parallel^2 + m^2[1+\sqrt{(1+2eB/m^2)}]^2$, respectively. These are the limiting photon energies for pair production by the differently polarized photons propagating through a magnetized vacuum, as given by Toll (1952) and Alder (1971).

5. An alternative method for calculating the vacuum polarization tensor

The following is an alternative method for calculating the vacuum polarization tensor based on a known expression for gyromagnetic emission.

The probability per unit time for emission of a photon in the mode σ in the range $\mathrm{d}^3\boldsymbol{k}/(2\pi)^3$ of wavenumbers by an electron with initial quantum numbers q and final quantum numbers q' is given by (Melrose 1974a)

$$w^{\sigma(\mathrm{em})}_{qq'}(\boldsymbol{k}) = \frac{8\pi^2 e^2 R^\sigma_E(\boldsymbol{k})}{|\omega^\sigma(\boldsymbol{k})|}|\boldsymbol{e}^{\sigma *}(\boldsymbol{k})\,.\,\boldsymbol{\Gamma}^{++}_{qq'}(\boldsymbol{k})|^2\delta(\omega^\sigma(\boldsymbol{k})-\varepsilon_q+\varepsilon_{q'})\delta(p'_z-p_z+k_\parallel), \tag{26}$$

with $R^\sigma_E(\boldsymbol{k})$ the ratio of electrical energy to total energy in the mode σ and with $\boldsymbol{\Gamma}^{++}_{qq'}(\boldsymbol{k})$ given by the 3-vector part of (5) with $\epsilon = \epsilon' = +1$. The probability per unit time for absorption of a photon in the mode σ is obtained from (26) by replacing $\boldsymbol{k}$ by $-\boldsymbol{k}$, using

$$\omega^\sigma(-\boldsymbol{k}) = -\omega^\sigma(\boldsymbol{k}), \qquad \boldsymbol{e}^\sigma(-\boldsymbol{k}) = \boldsymbol{e}^{\sigma *}(\boldsymbol{k}), \qquad R^\sigma_E(-\boldsymbol{k}) = R^\sigma_E(\boldsymbol{k}), \tag{27}$$

and interchanging quantum numbers q and q'. Detailed balancing then requires that one have, for $\epsilon = \epsilon' = 1$,

$$\boldsymbol{\Gamma}^{\epsilon\epsilon'}_{qq'}(-\boldsymbol{k}) = [\boldsymbol{\Gamma}^{\epsilon'\epsilon}_{q'q}(\boldsymbol{k})]^*, \tag{28}$$

at least to within a phase factor. The identity (7) allows one to prove (28) for arbitrary ϵ and ϵ', and it also implies

$$(\Gamma^{\epsilon\epsilon'}_{qq'}(-\boldsymbol{k}))^0 = [(\Gamma^{\epsilon'\epsilon}_{q'q}(\boldsymbol{k}))^0]^*. \tag{29}$$

The probability per unit time for pair annihilation into a photon in the range $\mathrm{d}^3\boldsymbol{k}/(2\pi)^3$ is related to (26) by a crossing symmetry. By suitable re-labelling (26) becomes

$$w^{\sigma(\mathrm{ann})}_{qq'}(\boldsymbol{k}) = \frac{8\pi^2 e^2 R^\sigma_E(\boldsymbol{k})}{|\omega^\sigma(\boldsymbol{k})|}|\boldsymbol{e}^{\sigma *}(\boldsymbol{k})\,.\,\boldsymbol{\Gamma}^{+-}_{qq''}(\boldsymbol{k})|^2\delta(|\omega^\sigma(\boldsymbol{k})|-\varepsilon_q-\varepsilon_{q'})\delta(p'_z+p_z-k_\parallel), \tag{30}$$

where the quantum numbers q'' are the same as q' but with the sign of p'_z reversed and where $\omega^\sigma(\boldsymbol{k})$ has been replaced by its modulus to avoid ambiguity.

For a vacuum containing a static magnetic field in the 3-direction the absorption coefficient (e-folding energy decay rate) for the mode σ is

$$\gamma^{\sigma}(\boldsymbol{k})=\frac{eB}{2\pi}\sum_{\substack{n,n'\\ s,s'}}\int_{-\infty}^{\infty}\frac{\mathrm{d}p_z}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}p_z'\, w_{qq'}^{\sigma(\mathrm{ann})}(\boldsymbol{k}). \tag{31}$$

This result is the same as that obtained by using the relation (Melrose 1974b)

$$\gamma^{\sigma}(\boldsymbol{k})=-2\mathrm{i}\omega^{\sigma}(\boldsymbol{k})R_E^{\sigma}(\boldsymbol{k})e_i^{\sigma *}(\boldsymbol{k})e_j^{\sigma}(\boldsymbol{k})\varepsilon_{ij}^{(\mathrm{a})}(\boldsymbol{k},\omega^{\sigma}(\boldsymbol{k})), \tag{32}$$

where $\varepsilon_{ij}^{(\mathrm{a})}(\boldsymbol{k},\omega)$ is related to $\alpha_{ij}^{(\mathrm{a})}(\boldsymbol{k},\omega)$ by equation (1) and the anti-Hermitian part of the vacuum polarization tensor is that obtained from (8).

The alternative method for calculating the vacuum polarization tensor for a magnetized vacuum is based on (30)–(32). One starts from the probability for pair production given by (30) and identifies $\varepsilon_{ij}^{(\mathrm{a})}(\boldsymbol{k},\omega)$ from the equality of (31) and (32). The Hermitian part of the vacuum polarization tensor may be calculated from this by using the (doubly subtracted) Kramers–Kronig relation (Lifshitz and Pitaevskii 1974, § 108). The vacuum polarization 4-tensor may then be obtained from the 3-tensor by using the requirements of charge continuity and gauge invariance (Melrose 1973).

The foregoing method is a generalization both of the method used by Toll (1952) and Erber (1966) to calculate the refractive index of a magnetized vacuum from the absorption coefficient for photo-pair production, and of the method used by Euwema and Wheeler (1956) and Lifshitz and Pitaevskii (1974, § 110) for an unmagnetized vacuum. This alternative method reproduces the result (24) for the anti-Hermitian part of the tensor.

Appendix 1. Summation formulae

The following identities are used elsewhere in this paper. For any function $f(n,n')$ of integers n and n', both positive semi-definite, one has

$$\begin{aligned}
&\sum_{n'=0}^{\infty}\sum_{n=0}^{\infty}f(n,n'+1)=\sum_{n'=0}^{\infty}\sum_{n=0}^{\infty}f(n+1,n'+1)+\sum_{n'=0}^{\infty}f(0,n'+1),\\
&\sum_{n'=0}^{\infty}\sum_{n=0}^{\infty}f(n+1,n')=\sum_{n'=0}^{\infty}\sum_{n=0}^{\infty}f(n+1,n'+1)+\sum_{n=0}^{\infty}f(n+1,0),
\end{aligned}\tag{A.1}$$

$$\sum_{n'=0}^{\infty}\sum_{n=0}^{\infty}f(n,n')=\sum_{n'=0}^{\infty}\sum_{n=0}^{\infty}f(n+1,n'+1)+\sum_{n=0}^{\infty}f(n+1,0)+\sum_{n'=0}^{\infty}f(0,n'+1)+f(0,0),$$

$$\begin{aligned}
&\sum_{n'=0}^{\infty}\sum_{n=0}^{\infty}f(n+1,n'+1)\delta_{n'-n,-1}=\sum_{n=0}^{\infty}f(n+2,n+1),\\
&\sum_{n'=0}^{\infty}\sum_{n=0}^{\infty}f(n+1,n'+1)\delta_{n'-n,1}=\sum_{n=0}^{\infty}f(n+1,n+2).
\end{aligned}\tag{A.2}$$

Appendix 2. Special cases of the Hermitian part

A.2.1. Zero-field limit

Except in the limit $k_{\perp}=0$ the zero-field vacuum polarization tensor is difficult to extract from (19). It may be found by using our method but with the electron propagator replaced by its zero-field expression. Alternatively, the method of Heitler (1954, § 32) may be used. Applying our renormalization procedure to the resulting unrenormalized expressions gives

$$\operatorname{Re} G_0 = \frac{e^2k^2}{4\pi^3 k_\perp}\int d^3p \frac{p_x(\varepsilon_p-\varepsilon_{p-k})-k_\perp\varepsilon_p}{[\omega^2-(\varepsilon_p-\varepsilon_{p-k})^2]\varepsilon_p\varepsilon_{p-k}},$$
$$\operatorname{Re} G_1 = \operatorname{Re} G_2 = 0, \tag{A.3}$$

with $\varepsilon_p := (m^2+|p|^2)^{1/2}$ and $\varepsilon_{p-k} := (m^2+|p-k|^2)^{1/2}$. Integrating around the poles in the usual way gives

$$\operatorname{Re} G_0 = -\frac{e^2k^2}{4\pi^2}\left(\frac{1}{9}-\frac{(1-\psi\cot\psi)(4m^2+2k^2)}{3k^2}\right), \tag{A.4}$$

with $\sin^2\psi := k^2/4m^2$. The renormalized vacuum polarization tensor obtained from (A.4) is just the usual zero-field tensor (Akhiezer and Berestetskii 1965, § 47.3).

A.2.2. Long-wavelength limit

In the long-wavelength limit ($k_\perp = k_\| = 0$) the limits

$$\lim_{x\to 0}\frac{J^n_{\nu+1}(x)J^{n+1}_\nu(x)}{x^{1/2}} = (n+1)^{1/2}(\delta_{\nu,0}-\delta_{\nu,-1}),$$
$$\lim_{x\to 0}\frac{J^n_\nu(x)J^{n+1}_{\nu-1}(x)}{x^{1/2}} = (n+1)^{1/2}(\delta_{\nu,-1}-\delta_{\nu,0}),$$
$$\lim_{x\to 0}\{[J^n_\nu(x)]^2+[J^{n+1}_\nu(x)]^2\} = 2\delta_{\nu,0}, \tag{A.5}$$
$$\lim_{k_\|\to 0}\frac{\varepsilon_{n+1}p'_z-\varepsilon'_{n+1}p_z}{k_\|} = -\frac{k_\|}{\varepsilon_{n+1}}(m^2+p^2_{n+1}),$$

where $\varepsilon'_n := [m^2+(p'_z)^2+p_n^2]^{1/2}$, and the identities (A.2) may be applied to (19) to give

$$\operatorname{Re} G_0 = -\frac{e^3B\omega^2}{2\pi}\int_{-\infty}^{\infty}\frac{dp_z}{2\pi}\sum_{n=0}^{\infty}\frac{2[1+(m^2+p_z^2)/(\varepsilon_n\varepsilon_{n+1})]}{[\omega^2-(\varepsilon_n+\varepsilon_{n+1})^2](\varepsilon_n+\varepsilon_{n+1})},$$
$$\operatorname{Re} G_1 = 0 \tag{A.6}$$
$$\operatorname{Re} G_2 = \frac{e^3B\omega^2}{2\pi}\int_{-\infty}^{\infty}\frac{dp_z}{2\pi}\sum_{n=0}^{\infty}\left(\frac{2[1+(m^2+p_z^2)/(\varepsilon_n\varepsilon_{n+1})]}{[\omega^2-(\varepsilon_n+\varepsilon_{n+1})^2](\varepsilon_n+\varepsilon_{n+1})}-\frac{a_n(m^2+p_n^2)}{(\omega^2-4\varepsilon_n^2)\varepsilon_n^3}\right).$$

In the limit $\omega^2 \ll 4m^2$ the integration over p_z in (A.6) is straightforward if one makes the expansion

$$\frac{1+(m^2+p_z^2)/(\varepsilon_n\varepsilon_{n+1})}{(\varepsilon_n+\varepsilon_{n+1})^3}$$
$$= \frac{1}{8\varepsilon_n^5}\left[(2\varepsilon_n^2-p_n^2)-\frac{1}{2}(8\varepsilon_n^2-5p_n^2)\left(\frac{eB}{\varepsilon_n^2}\right)+\frac{1}{4}(30\varepsilon_n^2-21p_n^2)\left(\frac{eB}{\varepsilon_n^2}\right)^2\right.$$
$$\left.-\frac{7}{2}(4\varepsilon_n^2-3p_n^2)\left(\frac{eB}{\varepsilon_n^2}\right)^3+\frac{15}{8}(14\varepsilon_n^2-11p_n^2)\left(\frac{eB}{\varepsilon_n^2}\right)^4+\ldots\right]. \tag{A.7}$$

This gives, with $L := eB/m^2$,

$$\operatorname{Re} G_0 = \frac{e^2\omega^2 L}{105(2\pi)^2} \sum_{n=0}^{\infty} F(nL),$$
$$\operatorname{Re} G_1 = 0, \tag{A.8}$$
$$\operatorname{Re} G_2 = -\frac{e^2\omega^2 L}{105(2\pi)^2} \sum_{n=0}^{\infty} (F(nL) - G(nL)),$$

where

$$F(x) := \frac{35(3+4x)}{(1+2x)^2} - \frac{140L(1+x)}{(1+2x)^3} + \frac{42L^2(5+4x)}{(1+2x)^4} - \frac{112L^3(3+2x)}{(1+2x)^5} + \frac{80L^4(7+4x)}{(1+2x)^6},$$
$$G(x) := \frac{35a_n}{(1+2x)}. \tag{A.9}$$

The sum over n in (A.8) may be expressed as an expansion in L by using Euler's summation formula

$$\sum_{n=0}^{\infty} F(nL) = \frac{1}{2}(F(\infty) + F(0)) + \frac{1}{L}\int_0^{\infty} \mathrm{d}x\, F(x)$$
$$+ \sum_{k=1}^{\infty} \frac{B_{2k} L^{2k-1}}{(2k)!} (F^{(2k-1)}(\infty) - F^{(2k-1)}(0)), \tag{A.10}$$

where the Bernoulli numbers are $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, etc (Abramowitz and Stegun 1965, p 806). In this way (A.8) with (A.9) becomes

$$\operatorname{Re} G_0 = \frac{e^2\omega^2}{3(2\pi)^2} \int_0^{\infty} \mathrm{d}x \frac{3+4x}{(1+2x)^2} - \frac{e^2\omega^2 L^2}{90\pi^2} + \frac{e^2\omega^2 L^4}{105\pi^2},$$
$$\operatorname{Re} G_1 = 0, \tag{A.11}$$
$$\operatorname{Re} G_2 = -\frac{e^2\omega^2}{3(2\pi)^2} \int_0^{\infty} \frac{\mathrm{d}x}{(1+2x)} + \frac{7e^2\omega^2 L^2}{180\pi^2} - \frac{13e^2\omega^2 L^4}{630\pi^2},$$

where only terms up to L^4 have been retained. Only even powers of the magnetic field appear in these expansions, as required by Furry's (1937) theorem. The first term in $\operatorname{Re} G_0$ is identical to the long-wavelength, low-frequency limit of (A.3). It is identified with the long-wavelength, low-frequency limit of (A.4) and is thus equal to $e^2\omega^2/60\pi^2 m^2$. The first term of $\operatorname{Re} G_2$ resulted from subtracting two divergent integrals. It is spurious, as can be seen from the zero-field limit of (A.6), and is to be ignored.

The expansion (A.11) agrees with that obtained from equation (36) of I. The resulting expression for $\varepsilon_{ij}(\mathbf{0}, \omega)$ obtained using (1) agrees with the dielectric tensor calculated from the Heisenberg and Euler (1936) effective Lagrangian. Keeping the next-order terms in k gives the magnetic permeability tensor. The equivalent dielectric tensor is related to the dielectric tensor $\tilde{\varepsilon}_{ij}$ (electric dipole approximation) and magnetic permeability tensor μ_{ij} (magnetic dipole approximation) by

$$\varepsilon_{ij}(\boldsymbol{k}, \omega) = \tilde{\varepsilon}_{ij} + \frac{|\boldsymbol{k}|}{\omega^2}[\delta_{ij} - \kappa_i\kappa_j + \delta_{ij}(\kappa_m\kappa_n\mu^{-1}_{mn} - \mu^{-1}_{ss})$$
$$+ \mu^{-1}_{ij} + \kappa_i\kappa_j\mu^{-1}_{ss} - \kappa_r\mu^{-1}_{ri}\kappa_j - \kappa_i\mu^{-1}_{js}\kappa_s], \tag{A.12}$$

with $\boldsymbol{\kappa} := \boldsymbol{k}/|\boldsymbol{k}|$ and with μ^{-1}_{ij} denoting the tensor inverse to μ_{ij}.

A.2.3. Strong-field limit

For low-energy photons ($\omega^2-k_\parallel^2, k_\perp^2 \ll 4m^2$) traversing a strong magnetic field $B \gg B_c$ ($:= m^2/e$), one has

$$\varepsilon_n^2 = m^2(1+2neB/B_c)+p_z^2 \gg \varepsilon_0^2 \qquad (n\neq 0),$$
$$(\varepsilon_n')^2 = m^2(1+2neB/B_c)+(p_z')^2 \gg (\varepsilon_0') \qquad (n\neq 0) \tag{A.13}$$

and

$$\varepsilon_0' \approx \varepsilon_0 - (p_z k_\parallel/\varepsilon_0).$$

The dominant term in (19) in this limit is therefore the term containing the factor $[\omega^2-(\varepsilon_0+\varepsilon_0')^2]^{-1}$. This gives, using $L_n^\nu(0)=(n+\nu)!/(n!\nu!)$,

$$\mathrm{Re}\, G_0 = \mathrm{Re}\, G_1 = 0,$$
$$\mathrm{Re}\, G_2 = \frac{e^3B(\omega^2-k_\parallel^2)}{2\pi k_\parallel}\int_{-\infty}^{\infty}\frac{\mathrm{d}p_z}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}p_z'}{2\pi}2\pi\delta(p_z'-p_z+k_\parallel)\frac{\varepsilon_0 p_z'-\varepsilon_0' p_z}{[\omega^2-(\varepsilon_0+\varepsilon_0')^2]\varepsilon_0\varepsilon_0'}. \tag{A.14}$$

After performing the p_z' and p_z integrals this gives

$$\mathrm{Re}\,\chi_2 := \frac{4\pi\,\mathrm{Re}\,G_2}{(\omega^2-k_\parallel^2)} \approx \frac{e^2}{3\pi}\frac{B}{B_c}. \tag{A.15}$$

This result agrees with equation (50) of I. The first part of equation (51) of I gives the refractive index of a magnetized vacuum with $1 \ll B/B_c \ll 3\pi/e^2$ which reproduces the dominant strong-field term of Tsai and Erber (1975). For $B/B_c \gg 3\pi/e^2$ the relevant refractive index is the second one in equation (51) of I, but for such strong magnetic fields it may be necessary to consider higher-order radiative corrections.

Appendix 3. Special cases of the anti-Hermitian part

A.3.1. Long-wavelength limit

For propagation along the magnetic field one sets $k_\perp=0$ in (24) and uses $L_n^\nu(0)=(n+\nu)!/(n!\nu!)$ to obtain

$$\alpha^{11(a)}(k_\parallel,\omega)=\alpha^{22(a)}(k_\parallel,\omega)$$
$$= -\frac{ie^3B}{2\pi}\mathrm{sgn}(\omega^2-k_\parallel^2)\,\mathrm{sgn}\,\omega\sum_{n=0}^{\infty}\left(\beta_1^- - \frac{2p_n^2}{(\omega^2-k_\parallel^2)}\right)\frac{\theta(\kappa_{1,n-1}^2)}{\kappa_{1,n-1}}$$
$$\alpha^{33(a)}(k_\parallel,\omega) = -\frac{ie^3B\omega^2}{\pi(\omega^2-k_\parallel^2)^2}\mathrm{sgn}(\omega^2-k_\parallel^2)\,\mathrm{sgn}\,\omega\sum_{n=0}^{\infty}a_n(m^2+p_n^2)\frac{\theta(\kappa_{0,n-1}^2)}{\kappa_{0,n-1}}. \tag{A.16}$$

For an electron gas Svetozarova and Tsytovich (1962) calculated the anti-Hermitian part of the tensor in this limit but they ignored the vacuum terms given above.

In the long-wavelength limit, $k_\parallel=0$ in (A.16) gives

$$\alpha^{11(a)}(0,\omega)=\alpha^{22(a)}(0,\omega) = -\frac{ie^3B}{2\pi\omega}\sum_{n=0}^{n_1}\frac{\omega^2-2eB(1+2n)}{\{\omega^2[1-(2eB/\omega^2)]^2-4(m^2+p_n^2)\}^{1/2}},$$
$$\alpha^{33(a)}(0,\omega) = -\frac{ie^3B}{\pi\omega}\sum_{n=0}^{n_2}\frac{a_n(m^2+p_n^2)}{[\omega^2-4(m^2+p_n^2)]^{1/2}}. \tag{A.17}$$

with n_1 and n_2 such that $\{\omega^2[1-(2eB/\omega^2)]^2-4m^2\}/8eB$ and $(\omega^2-4m^2)/8eB$ are greater than or equal to n_1 and n_2, respectively, and less than n_1+1 and n_2+1 respectively. Cover and Kalman (1974) obtained the opposite sign for $\alpha^{33(a)}(0, \omega)$, but this is probably only a difference of notation.

A.3.2. Zero-field limit

In the zero-field limit one sets $k_\parallel = |\boldsymbol{k}|$ in (A.16) and makes the replacements

$$p_n^2 \to p_\perp^2,$$
$$\kappa_{\nu,n} \to \kappa := \left(1-\frac{4(m^2+p_\perp^2)}{\omega^2-|\boldsymbol{k}|^2}\right)^{1/2},$$
$$\sum_{n=0}^{\infty} \theta(\kappa_{\nu,n}^2) \to \int_0^{p_{\perp 0}} \frac{\mathrm{d}p_\perp p_\perp}{eB}\, \theta(p_{\perp 0}^2),$$

with $p_{\perp 0} := [\frac{1}{4}(\omega^2-|\boldsymbol{k}|^2)-m^2]^{1/2}$. On performing the integration one obtains the zero-field tensor

$$\begin{aligned}\alpha_0^{11(a)}(|\boldsymbol{k}|, \omega) &= \alpha_0^{22(a)}(|\boldsymbol{k}|, \omega)\\ &= -\frac{\mathrm{i}e^2 \operatorname{sgn}\omega}{6\pi}[m^2+\tfrac{1}{2}(\omega^2-|\boldsymbol{k}|^2)]\left(1-\frac{4m^2}{(\omega^2-|\boldsymbol{k}|^2)}\right)^{1/2}\theta(\omega^2-|\boldsymbol{k}|^2-4m^2),\\ \alpha_0^{33(a)}(|\boldsymbol{k}|, \omega) &= \frac{\omega^2}{(\omega^2-|\boldsymbol{k}|^2)}\alpha_0^{11(a)}(k_\parallel, \omega).\end{aligned} \tag{A.18}$$

This result agrees with that of Feynman (1949) and corrects that of Tsytovich (1961) by a factor of 2.

References

Abramowitz M and Stegun I A 1965 *Handbook of Mathematical Functions* (New York: Dover)
Adler S L 1971 *Ann. Phys., NY* **67** 599
Akhiezer A I and Berestetskii V B 1965 *Quantum Electrodynamics* (New York: Interscience)
Berestetskii V B, Lifshitz E M and Pitaevskii L P 1971 *Relativistic Quantum Theory* part 1 (Oxford: Pergamon)
Constantinescu D H 1972a *Nucl. Phys.* B **36** 121
—— 1972b *Lett. Nuovo Cim.* **5** 766
Cover R A and Kalman G 1974 *Phys. Rev. Lett.* **33** 1113
Erber T 1966 *Rev. Mod. Phys.* **38** 626
Euwema R N and Wheeler J A 1956 *Phys. Rev.* **103** 803
Feynman R P 1949 *Phys. Rev.* **76** 769
Furry W H 1937 *Phys. Rev.* **51** 125
Géhéniau J 1950 *Physica* **16** 822
Géhéniau J and Demeur M 1951 *Physica* **17** 71
Gradsteyn I S and Ryzhik I M 1965 *Tables of Integrals, Series and Products* (New York: Academic)
Heisenberg W and Euler H 1936 *Z. Phys.* **98** 714
Heitler W 1954 *Quantum Theory of Radiation* 3rd edn (London: Oxford University Press)
Lifshitz E M and Pitaevskii L P 1974 *Relativistic Quantum Theory* part 2 (Oxford: Pergamon)
Melrose D B 1973 *Plasma Phys.* **15** 99
—— 1974a *Plasma Phys.* **16** 845
—— 1974b *Aust. J. Phys.* **27** 279

Melrose D B and Stoneham R J 1976 *Nuovo Cim.* A **32** 435
Montgomery D C and Tidman D A 1964 *Plasma Kinetic Theory* (New York: McGraw-Hill)
Shabad A E 1975 *Ann. Phys., NY* **90** 166
Svetozarova G I and Tsytovich V N 1962 *Izv. Vyssh. Ucheb. Zaved. Radiofiz.* **5** 658
Toll J S 1952 *PhD Thesis*, Princeton University, University Microfilm DS-11044
Tsai W and Erber T 1974 *Phys. Rev.* D **10** 492
—— 1975 *Phys. Rev.* D **12** 1132
Tsytovich V N 1961 *Sov. Phys.-JETP* **13** 1249